1 Sranda potential

1.1 Level spectrum and Level dynamics

The Hamiltonian

$$H = T + V = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(x^2 - 1\right)^2 + Ax + Dy^2 \tag{1}$$

of the Sranda system is separable in x and y coordinates. In the x coordinate it is equivalent with the CUSP 1D system

$$H_x = \frac{1}{2}p_x^2 + x^4 + Ax + Bx^2 + C \tag{2}$$

with $B = -2$, $C = 1$, in the y coordinate it is a simple harmonic oscillator

$$H_y = \frac{1}{2}p_y^2 + Dy^2. \tag{3}$$

The quantum energy spectrum of the Sranda system is constructed in the following way: First the spectrum of the CUSP system is calculated, see Figure 1. Second one builds the equidistant spectrum of the harmonic oscillator

$$E_i^{(\mathrm{HO})} = \hbar\omega^{(\mathrm{HO})}\left(i + \frac{1}{2}\right), \tag{4}$$

where $\omega^{(\mathrm{HO})} = \sqrt{2D}$ (note that this spectrum does not depend on the control parameter A). Finally, the CUSP spectrum is copied with each copy shifted by the value $E_i^{(\mathrm{HO})}$, $i = 0, 1, 2, \ldots$ and superimposed to build the final spectrum. Examples for various values of D are presented in Figures 2–3.

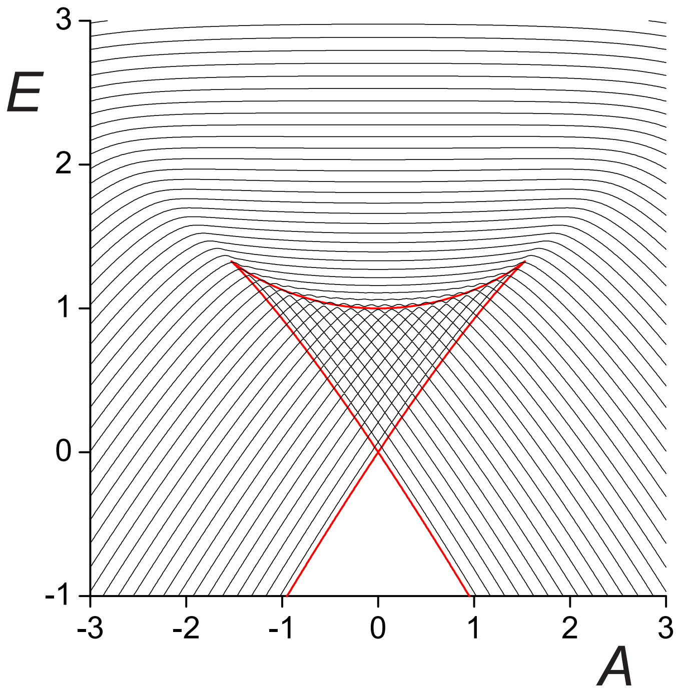

Figure 1: Level dynamics of the CUSP system (2) with $\hbar = 0.05$.

Note that the frequency in (each) minimum of the CUSP part is $\omega_0^{(\mathrm{CUSP})} = 2\sqrt{2}$ in the symmetric $A = 0$ case and it increases (decreases) slightly when the minimum becomes deeper (shallower) with

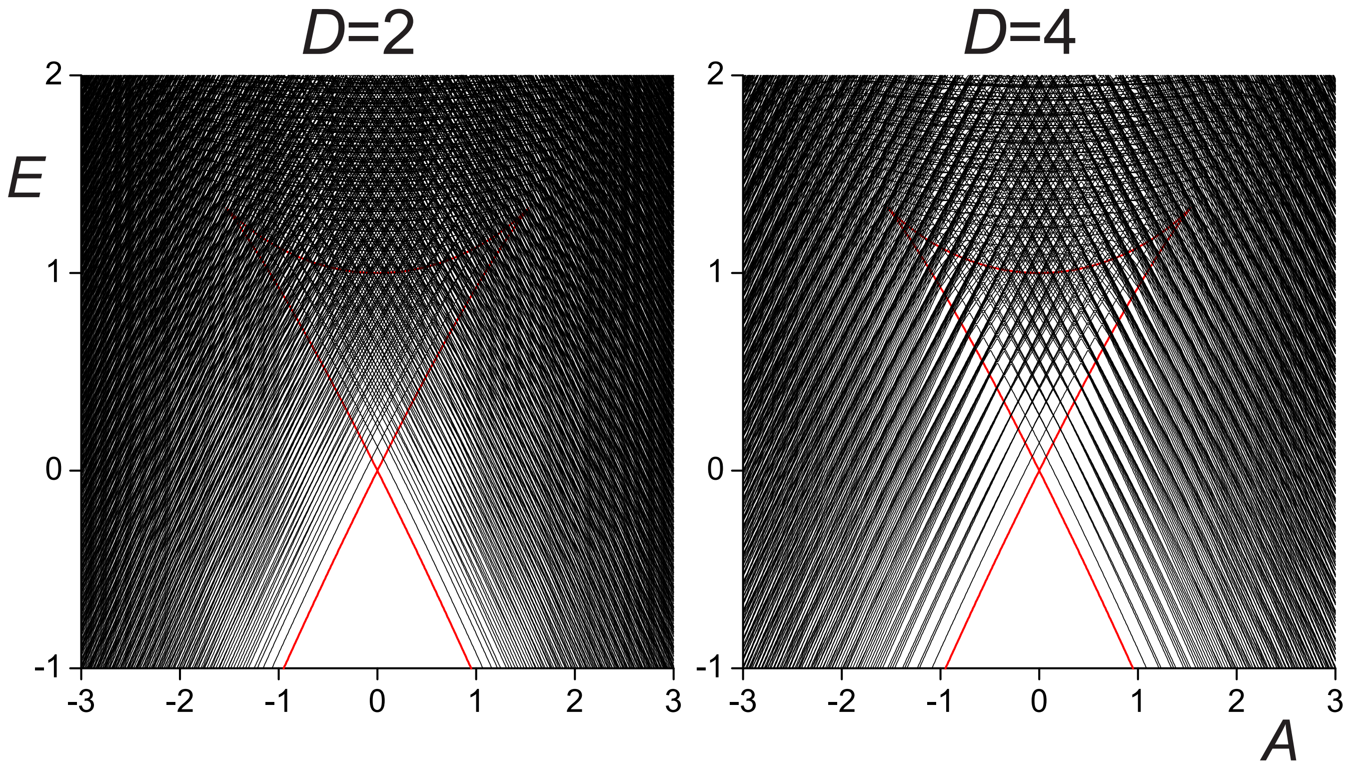

Figure 2: Level dynamics of the Sranda version of the Creagh-Whelan system (1) with $\hbar = 0.05$ and various values of the parameter D. The thicker red curves correspond with the position of local extreemes of the potential and surrounds the critical triangle.

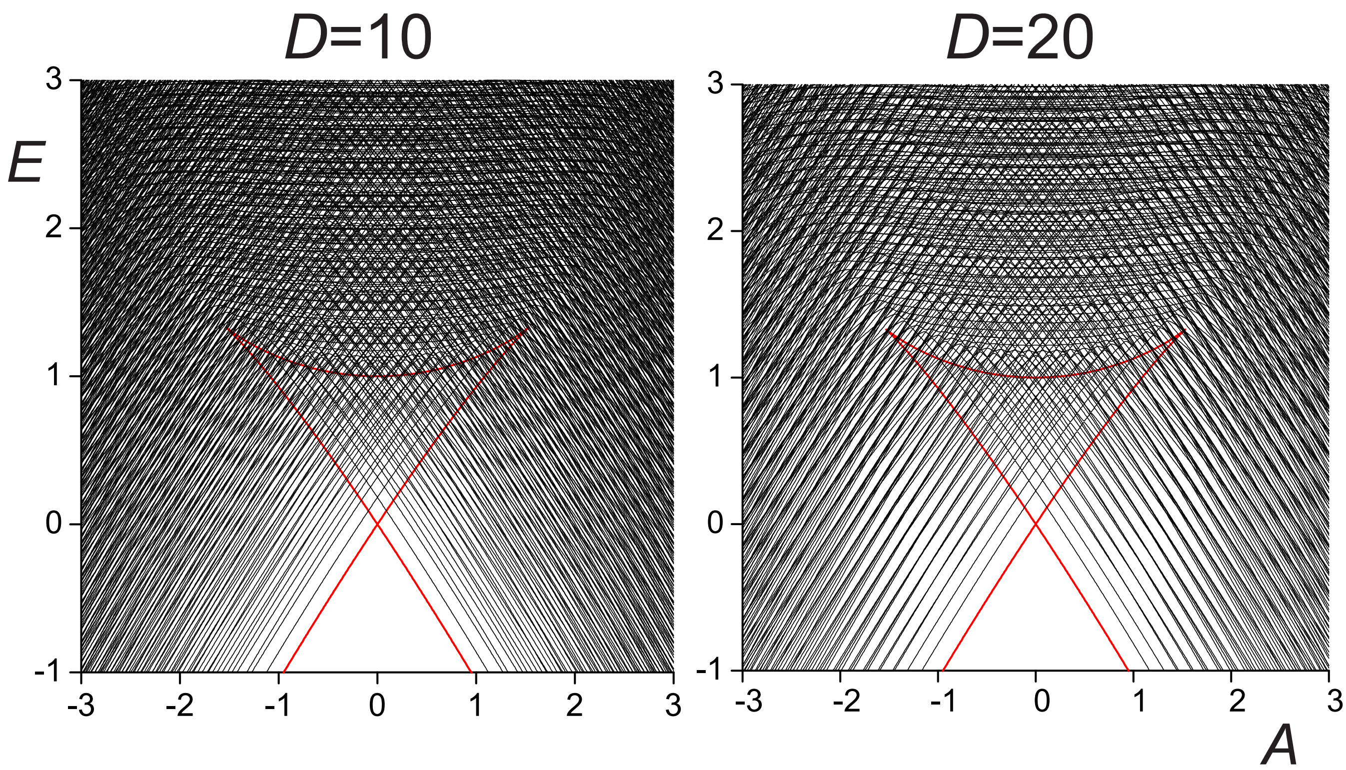

Figure 3: Continuation of Figure 2.

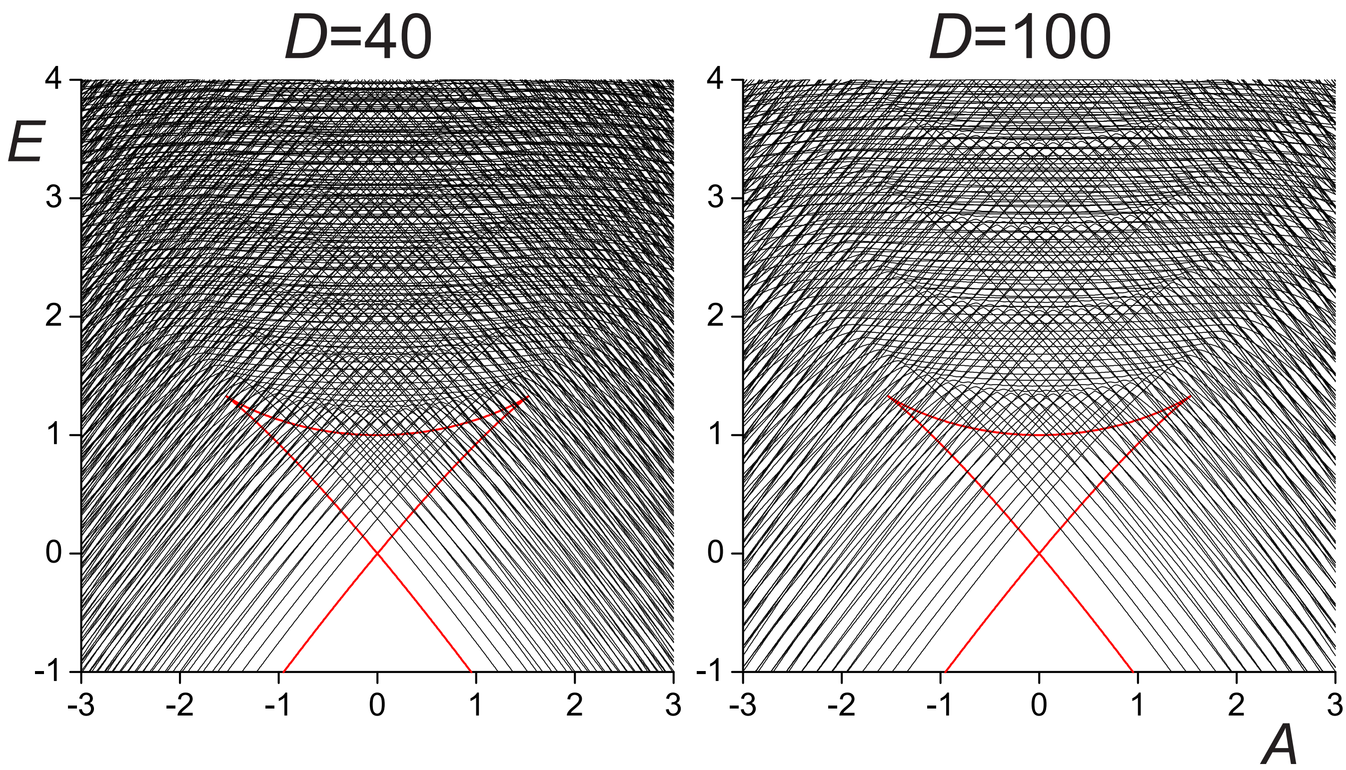

Figure 4: Continuation of Figure 2.

varying A, see Figure 5. Therefore, one can expect that the critical triangles connected with each of the copy of the CUSP could be easily distinguishable when either $D \ll 4$ (y-soft case) or $D \gg 4$ (y-rigid case).

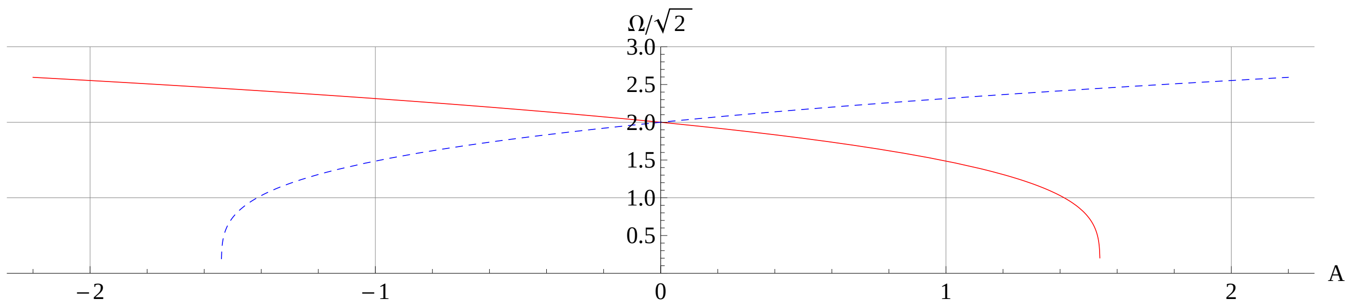

Figure 5: Frequency in the $x > 0$ minimum (red solid curve) and in the $x < 0$ minimum (blue dashed curve). The minima disappear at the spinodal points $A = \pm\sqrt{64/27}$.

1.2 Smoothed level density

The level density

$$\rho_\sigma(E) = \sum_i \delta_\sigma(E - E_i) \tag{5}$$

is smoothed using the Gaussian function with zero mean and with and with:

1. Constant variance σ

$$\delta_\sigma = \frac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}}. \tag{6}$$

2. Variance proportional to the reciprocal smooth level density $\sigma_i = \frac{\sigma_0}{\bar{\rho}(E_i)}$

$$\delta_{i;\sigma_0} = \frac{1}{\sqrt{2\pi\sigma_i^2}} \mathrm{e}^{-\frac{x^2}{2\sigma_i^2}}. \tag{7}$$

Figure 6 shows the constant-σ version of the level density $\rho_\sigma(E)$ (first column), its first derivate $\mathrm{d}\rho_\sigma(E)/\mathrm{d}E$ (second column), and the level density subtracted by the smooth part of the level density $\bar{\rho}(E)$ calculated with the semiclassical formula (third column) for two extreme values of the parameter D (y-soft case $D = 0.5$ in the first row, y-rigid case $D = 100$ in the second row). The value of the parameter σ is chosen by eye so that the structure of the critical triangle can be distinguished the best. One sees that while the level density itself is too smooth to observe any structure, the derivative and the oscillating remnant show clearly the pattern of the critical triangles.

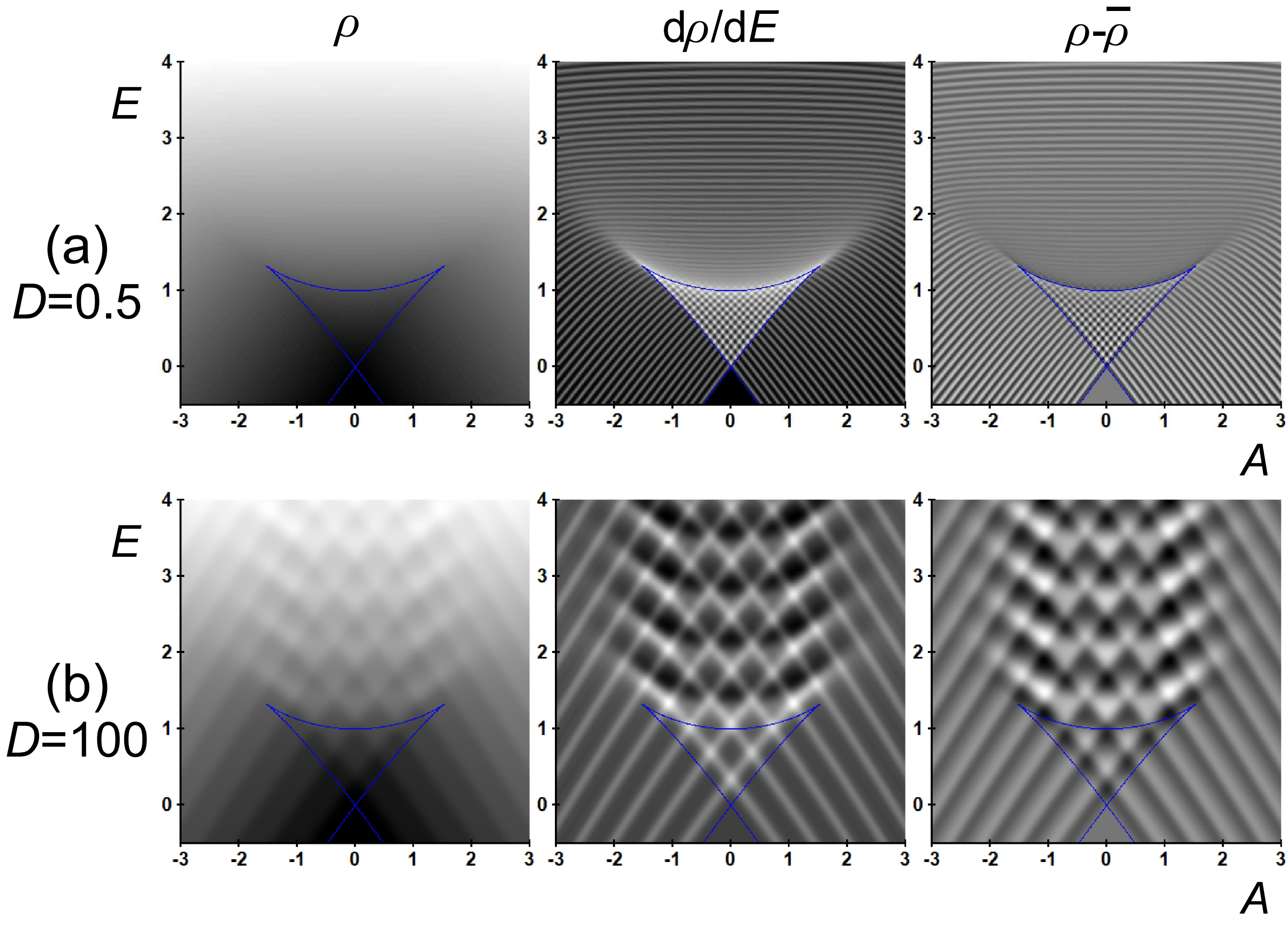

Figure 6: Smoothed level density of the Sranda potential for (a) $D = 0.5$ (y-soft case, $\sigma = 0.03$), and (b) $D = 100$ (y-rigid case, $\sigma = 0.11$). The first column shows the level density itself, the second its derivative, and the third the oscillating part around the smooth semiclassical value. Blue lines indicate the positions of the stationary points of the potential.

However, this is not true in the intermediate region where $\omega^{(\mathrm{HO})}$ is of the same order as the CUSP frequency $\omega^{(\mathrm{CUSP})} \approx 2\sqrt{2}$, or in other words, when the x and y level densities have simmilar values. This can be seen in Figure 7, especially in panels (b)–(d). The CUSP level level density for three different values of A and the HO level density $\rho^{(\mathrm{HO})} = 2/(\hbar\Omega^{(\mathrm{HO})})$ for various D are plotted in Figure 8.

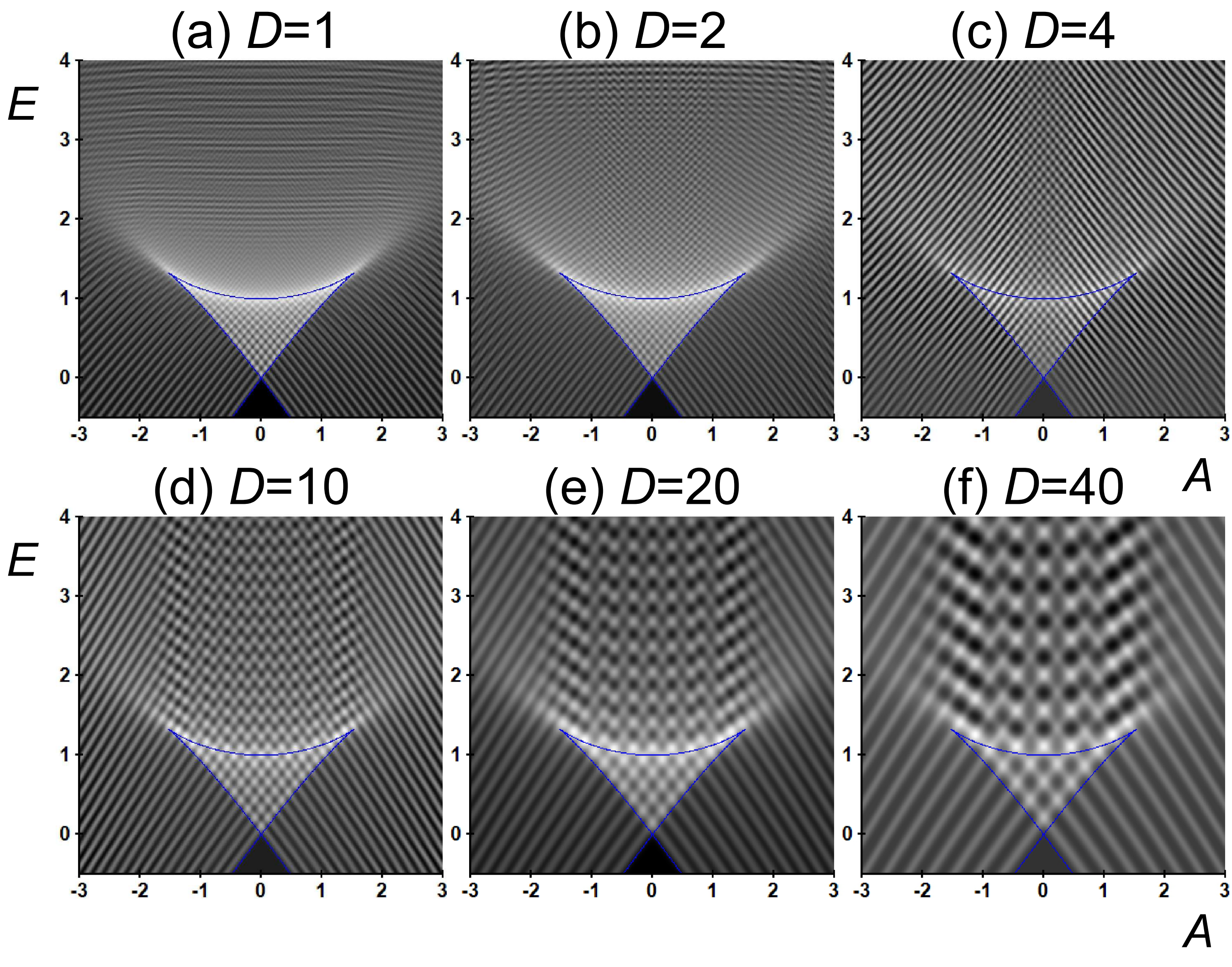

Figure 7: Energy derivative of the smoothed level density of the Sranda potential for various values of D. Variance σ used for each D is given in Table 1.2. Blue lines indicate the positions of the stationary points of the potential.

The optimal values of the variance σ used for the smoothing are given in Table 1.2. Note that the best value is $\sigma \approx 0.2/\sqrt{\rho^{(\mathrm{HO})}}$ (I don't understand why the square root). This table also gathers the values of the frequency in the y direction and of the correspondind level density.

The method with *rho*-dependent variance does not work well. The first reason is that at low energies the density is small, which results in rather high variance σ related with slow oscillations in the smoothed level density. The amplitude of this oscillations is rather high and exceeds the amplitude of the oscillations due to the critical triangles. The second reason is that the density of the critical triangles is constant, corresponding effectively with the level density of a 1D system, while the full level density is of a 2D system and it grows with energy. I do not put any example of this, but I recommend to check the following web pages where a full numerical study of the cases given in Figure 6 are given, including the dependence on the σ or σ_0 parameter: $D = 0.5$, $D = 100$.

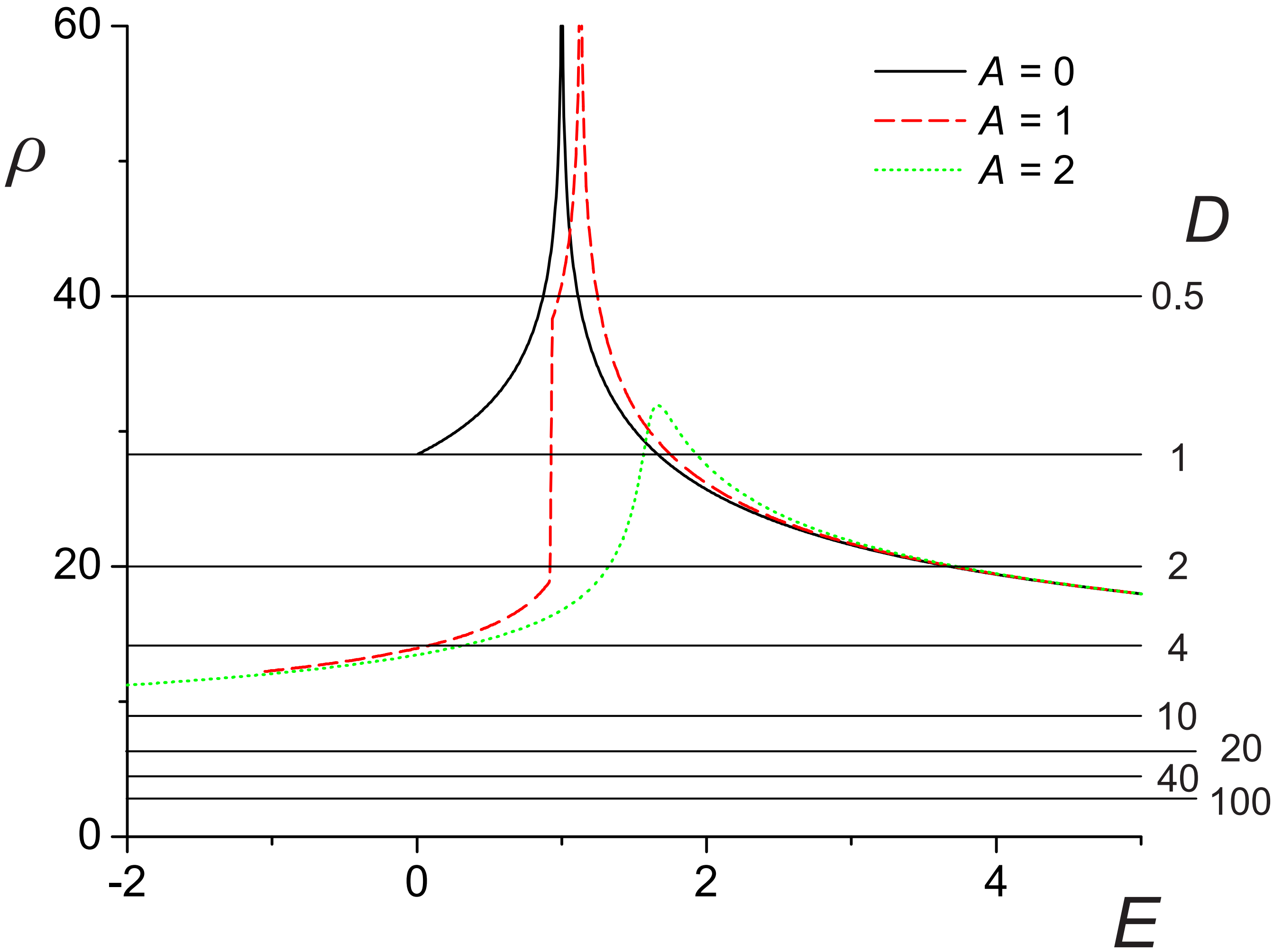

Figure 8: Level density of the CUSP part (thick black line for $A = 0$, red dashed line for $A = 1$, and green dotted line for $A = 2$) and the oscillator part (horizontal black lines) of the Sranda system with $\hbar = 0.05$.

2 Full Creagh-Whelan potential and the effect of chaos

In this section the full Creagh-Whelan Hamiltonian

$$H = T + V = \frac{1}{2}\left(p_x^2 + p_y^2\right) + \left(x^2 - 1\right)^2 + Ax + Bxy^2 + Cx^2y^2 + Dy^2 \tag{8}$$

D	$\omega^{(\mathrm{HO})}/\sqrt{2}$	$\rho^{(\mathrm{HO})}$	σ
0.5	0.71	0.03	40
1	1	0.04	28.3
2	1.41	0.05	20
4	2	0.06	14.1
10	3.16	0.07	8.94
20	4.47	0.09	6.32
40	6.32	0.10	4.47
100	10	0.12	2.83

Table 1: Values of the frequency and the level density of the harmonic oscillator part of the Sranda potential, and the variance σ used for smoothing in Figures 6–7.

is considered. In the following two subsections we shall restrict ourselves only on the case $B = 0$ with $A \leftrightarrow -A$.

2.1 Rigid case $C + D = 40$

The condition $C + D = \text{const}$ guarantees that the local behaviour of the potential at low energies remains the same when C and D are varied. However, with increasing C the system is generally more and more chaotic. This allows one to study the effect of chaos, departing from the fully regular separable case $C = 0$ to a highly caotic $D \approx 0$. Note that D must remain positive in order to keep the potential confining.

The first derivative of the smoothed level density is present in Figure 9 (and the corresponding separable case in Figure 7(f)). The chaoticity rises from left to right, which is proven by the classical fraction of regularity f_{reg} calculated from the Poincaré sections $y = 0$. It is clearly observed that the chaos destroys the regular patterns connected with superimposed critical triangle due to strong mixing between them.

2.2 Intermediate case $B + C = 4$

This configuration is shown in Figure 10. Unlike the situation displayed in Figure 7(c), the Planck constant here is bigger ($\hbar = 0.1$) in order to decrease the level density and thus reduce the computational time. First of all, the calculations demonstrate that there is a strong “interference” between the level density comming from the CUSP degree of freedom and the harmonic oscillator degree of freedom, which both have the same level density near the local minima of the potential. Second and more importantly for this study, increasing chaoticity smears again the clear periodic structure caused by formely separable critical triangles.

2.3 Asymmetric case $C = D = 40$

This case comes from the configuration shown in Figure 9(a). Unlike the previous cases the parameter B is varied here, which gradually introduces an asymmetry between the “volumes” of both potential wells. On top of that, higher B means also more chaos. The results are captured in Figure 11. The right well (more pronounced for $A < 0$) becomes narrower in the y direction—therefore more rigid—by increasing B, while the left well (dominant for $A > 0$) turns wider, softer, and having higher level density. Since the asymmetry prevents the accidental interference of the levels, the critical triangles are nicely visible (see for example panel (c)). Note that strongly asymmetric cases are quite challenging to calculate due to the fact that the mirror configurations A and $-A$ have

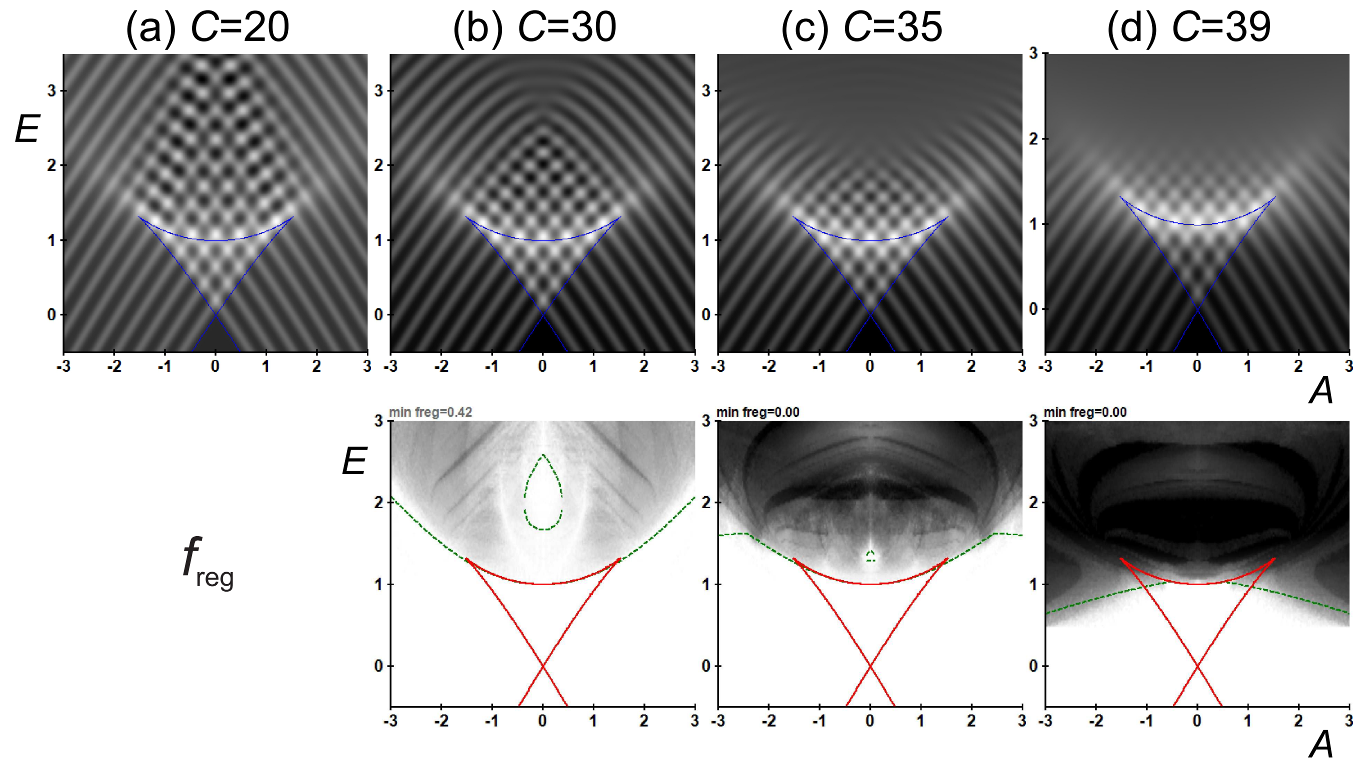

Figure 9: First row: Energy derivative of the smoothed level density of the Creagh-Whelan potential with $B = 0$, $C + D = 40$ (only the value of parameter C is given in the labels) and $\hbar = 0.05$. Variance $\sigma = 0.10$ is the same in all panels. Blue lines indicate the positions of the stationary points of the potential. Second row: Classical fraction of regularity (white shade—regular, black shad—chaotic). Red lines indicate the positions of the stationary points of the potential. Green dashed curves marks the convex-concave(-convex) transition of the kinematically accessible border calculated with the Geometry method (discussed elsewhere).

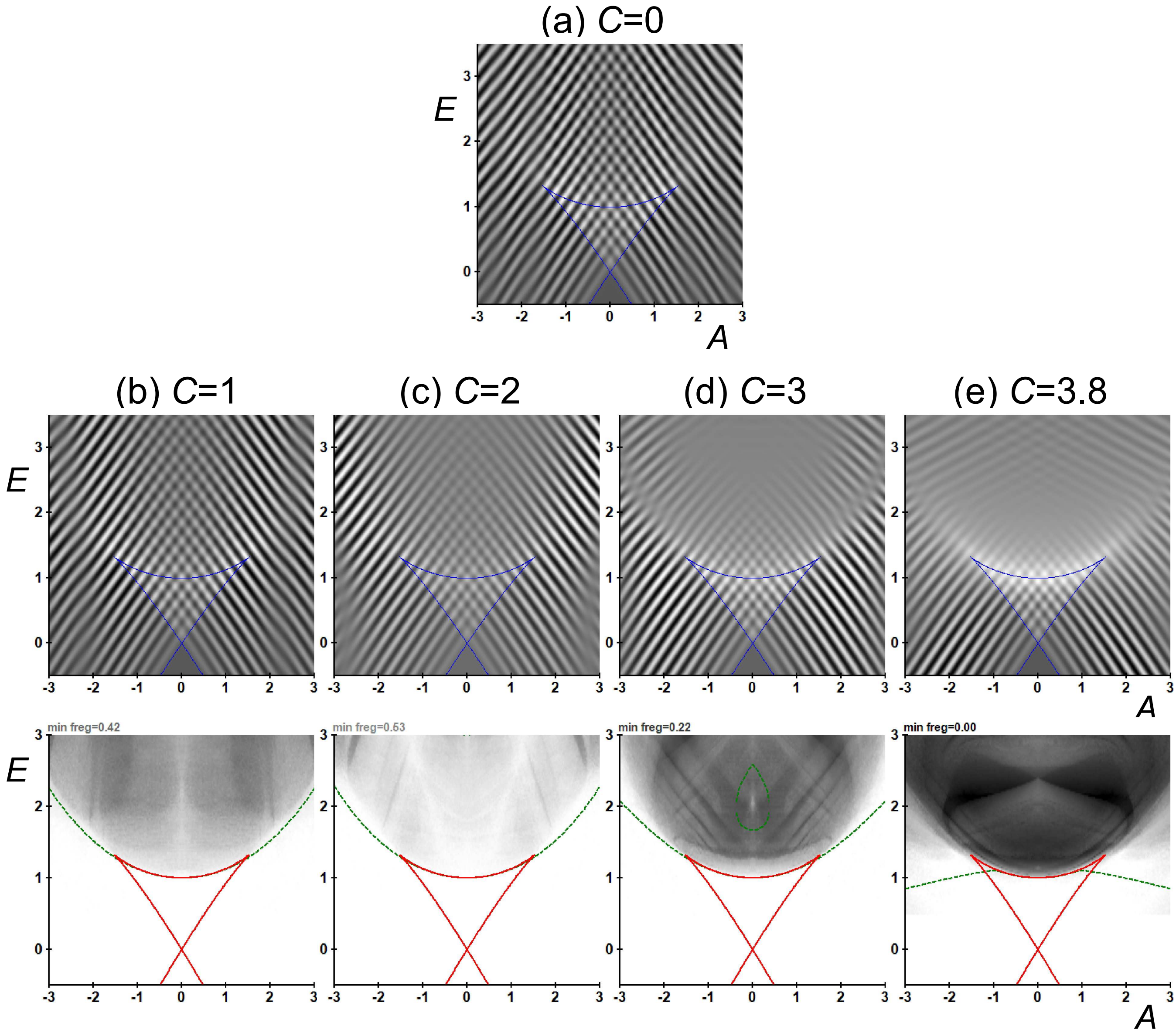

Figure 10: The same as in Figure 9, but for different values of the model's parameters: $B = 0$, $C + D = 4$, $\hbar = 0.1$.

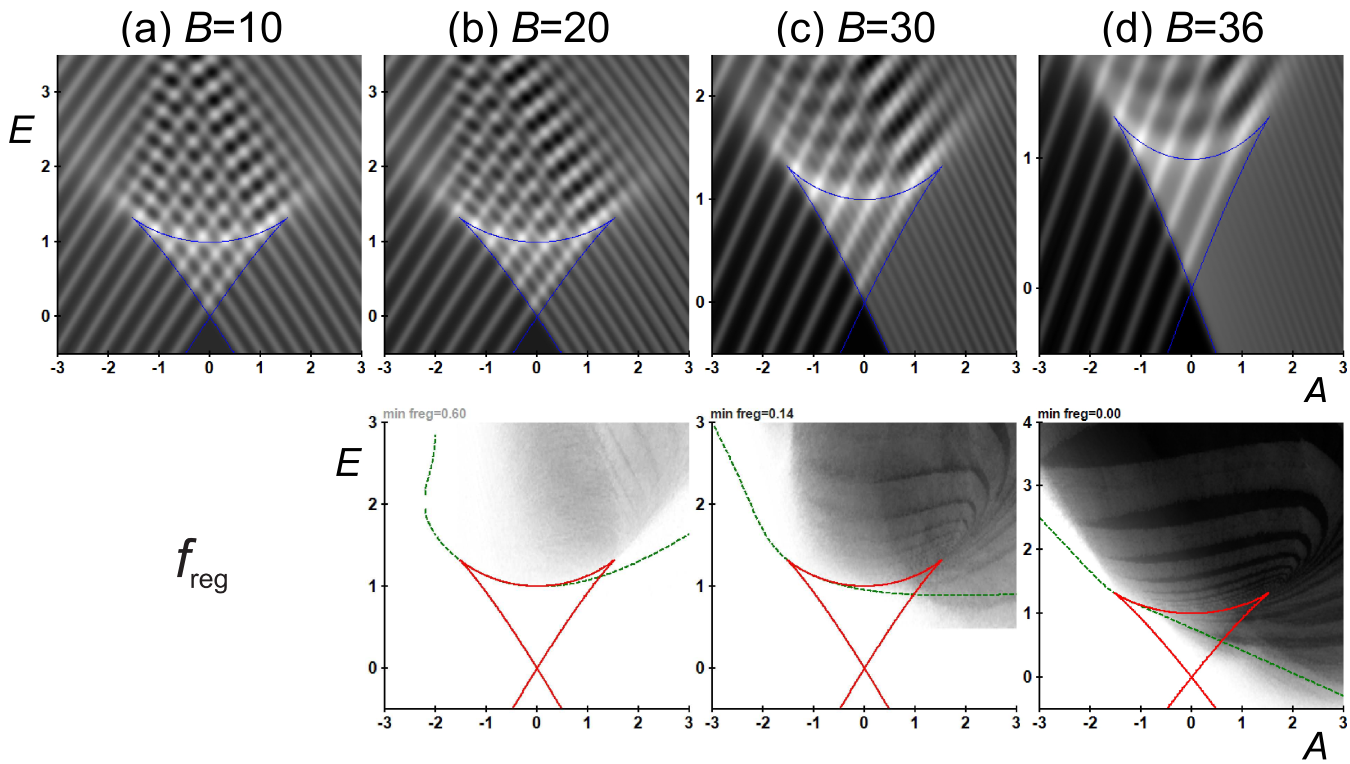

Figure 11: The same as in Figure 9, but for the asymmetric case: $C = D = 20$, $\hbar = 0.05$ and varying B.

very different level densities. There is not much space for the optimal balance of the required level density (governing the amount of details) and the total computed number of levels that determines the energy range. This is the reason why the energy range of the smoothed level density in panels (c)–(d) is smaller, despite the fact that the total number of levels is kept the same (500).